TENT-POLE MODEL OF THE FUTURE EURODOLLAR RATE

MARK FISHER

Abstract. This paper describes a simple model of the future Eurodollar rate that can be used to extract the probability of future FOMC target-rate ranges from the prices of Eurodollar futures and options. I call it the *tent-pole model*. Each future target-rate range has its own "tent" supported by a "pole" located at the expected future average overnight rate (conditioned on that target-rate range and adjusted to allow for the expected future term premium). The height of the pole is determined by the probability of the target-rate range. The tent itself is the continuous distribution of the future term premium around its expectation. The implied distribution for the future Eurodollar spot rate is a mixture of these tents. For near-term contracts, the tents will be narrow in width (since the uncertainty about the future term premium will be small) and consequently the mixture distribution will display distinct multi-modality. For longer-term contracts, the multi-modality will be obscured by blurring produced as a result of the greater uncertainty about the future term premium.

The approach to estimation is Bayesian. The locations, heights, and widths of the tents are all unobserved. A prior is used to associate expected future rates with their target-rate ranges. The empirical results are consistent with those based on a less parsimonious (and less interpretable) Bayesian nonparametric model.

Date: May 15, 2017 @ 10:53. *Filename*: "tent pole".

The views expressed herein are the author's and do not necessarily reflect those of the Federal Reserve Bank of Atlanta or the Federal Reserve System.

1. Introduction

This paper describes a simple model of the future Eurodollar rate that can be used to extract the probability of future FOMC target-rate ranges from the prices of Eurodollar futures and options. I call it the *tent-pole model*. Each future target-rate range has its own "tent" supported by a "pole" located at the expected future average overnight rate (conditioned on that target-rate range and adjusted to allow for the expected future term premium). The height of the pole is determined by the probability of the target-rate range. The tent itself is the continuous distribution of the future term premium around its expectation. The implied distribution for the future Eurodollar spot rate is a mixture of these tents. For near-term contracts, the tents will be narrow in width (since the uncertainty about the future term premium will be small) and consequently the mixture distribution will display distinct multi-modality. For longer-term contracts, the multi-modality will be obscured by blurring produced as a result of the greater uncertainty about the future term premium.

The approach to estimation is Bayesian. The locations, heights, and widths of the tents are all unobserved. A prior is used to associate expected future rates with their target-rate ranges. The empirical results are consistent with those based on a less parsimonious (and less interpretable) Bayesian nonparametric model.

The model involves a number of simplifications and approximations. Perhaps most importantly, the model ignores the differences between (*i*) the physical measure, (*ii*) the forward measure, and (*iii*) the futures measure (the latter two being "equivalent martingale measures").

2. The tent-pole model

The tent-pole model is a model of the distribution of the future Eurodollar spot rate (not to be confused with the Eurodollar *futures rate* which is discussed below). The distribution depends in part on the target-rate range set by the Federal Open Market Committee of the Federal Reserve System (FOMC). The FOMC meets periodically, eight times per year. At the end of each FOMC meeting a range for the target rate is announced (at about 2:30 PM local time in Washington, DC). The target ranges are 25 basis points wide.

Let r denote the future Eurodollar rate. It can be expressed as the sum of two components:

$$r = \mathcal{R} + \pi, \tag{2.1}$$

where $\mathcal{R}$ is the future expected average overnight rate and π is the future term premium. Each of the two components contributes to the uncertainty of the future Eurodollar rate. The future expected average overnight rate $\mathcal{R}$ depends on the future target-rate range. Let x_j denote the current expectation of $\mathcal{R}$ conditional on target-rate range j and let c_j denote the current probability of target-rate range j. Given this information, $\mathcal{R}$ has the following discrete distribution:

$$p(\mathcal{R} = x_j) = c_j. \tag{2.2}$$

By contrast, uncertainty about the future term premium π is given by a continuous distribution, which does not depend on the target-rate range:

$$p(\pi) = \mathsf{N}(\pi|\xi, \sigma^2). \tag{2.3}$$

Combining these two sources of uncertainty (which are assumed to be independent), the distribution for the future Eurodollar spot rate is given by the tent-pole model:

$$p(r) = \sum_{j=1}^{J} c_j \, \mathsf{N}(r|x_j + \xi, \sigma^2), \tag{2.4}$$

where there are J possible target rate ranges. In this model x_j determines the location of the tent pole associated with target-rate range j, c_j determines the height of tent pole j, and σ determines the width of the tents that are draped over the poles.

The prices of Eurodollar derivative securities depend on the distribution for the future Eurodollar rate (2.4). Let h denote the Eurodollar futures rate. The futures rate equals the expected future Eurodollar rate:

$$h = E[r] = \sum_{j=1}^{J} c_j \, (x_j + \xi). \tag{2.5}$$

Let z_i denote the value of an option on the future Eurodollar rate, where κ_i denotes the strike price for option i, and let $\gamma_i = 0$ indicate a put option and $\gamma_i = 1$ indicate a call option. The value of an option equals the discounted value of the expected payoff:

$$z_i = B E[g(r; \kappa_i, \gamma_i)] = B \sum_{j=1}^{J} c_j \, V(\kappa_i; \gamma_i, x_j + \xi, \sigma), \tag{2.6}$$

where B is the discount factor, $g(r; \kappa, \gamma)$ is the option payoff function [see (A.1)], and $V(\kappa; \gamma, x, \sigma)$ is expected value of the payoff [see (A.3)].

3. Inference

The goal is to use the tent-pole model to infer the probabilities of future target-rate ranges from the current prices of Eurodollar derivatives. The approach to inference is Bayesian.

Prior for unobserved parameters. Regarding the tent-pole model (2.4), the econometrician observes ξ (via the swaps market) but not $\theta = (c, x, \sigma)$, where $c = \{c_j\}_{j=1}^{J}$ and $x = \{x_j\}_{j=1}^{J}$. (The parameter of interest is c, while x and σ are nuisance parameters.)

Assume prior independence:

$$p(\theta) = p(c) \, p(x) \, p(\sigma). \tag{3.1}$$

Let the prior for x be given by

$$p(x) = \prod_{j=1}^{J} \mathsf{N}(x_j|\mu_j, \tau^2) \qquad \text{subject to } x_j < x_{j+1} \tag{3.2}$$

for $1 \le j < J$, where μ_j is the center of target-rate range j. For additional discussion of the prior for x see the appendix. Let the prior for σ be

$$p(\sigma) = \mathsf{SM}(\sigma|A) = \mathsf{SM}(\sigma|1/2, 2, A/\sqrt{3}) = \frac{(3/A^2)\,\sigma}{(1 + (3/A^2)\,\sigma^2)^{2/3}}, \tag{3.3}$$

where SM stands for the Singh–Maddala distribution.[1] Let the prior for $c \in \Delta^{J-1}$ be

$$p(c) = \mathsf{Dirichlet}(c|\alpha\,\underline{c}), \tag{3.4}$$

where $\underline{c} \in \Delta^{J-1}$ is the prior mean and $\alpha > 0$ is the concentration parameter.[2]

In summary, the prior depends on the following parameters: $(\tau, A, \alpha, \underline{c})$.

Likelihood and posterior. Let the *observed* Eurodollar futures rates and option values be given by

$$\mathcal{H} = h + u \tag{3.5a}$$

$$\mathcal{Z}_i = z_i + v_i, \tag{3.5b}$$

where $u \sim \mathsf{N}(0, \eta^2/\lambda)$ and $v_i \overset{\text{iid}}{\sim} \mathsf{N}(0, \eta^2)$ are measurement errors and λ is the precision of the measurement error for futures relative to that for options.[3]

Let $Y = (\mathcal{H}, \mathcal{Z}_1, \ldots, \mathcal{Z}_n)$. The likelihood is

$$p(Y|\theta, \eta) = \mathsf{N}(\mathcal{H}|h, \eta^2/\lambda) \prod_{i=1}^{n} \mathsf{N}(\mathcal{Z}_i|z_i, \eta^2). \tag{3.6}$$

Assume $p(\eta) \propto 1/\eta$ and integrate out η, producing

$$p(Y|\theta) = \int \frac{p(Y|\theta, \eta)}{\eta}\, d\eta \propto \lambda^{1/2} \Big(\lambda\,(\mathcal{H} - h)^2 + \sum_{i=1}^{n} (\mathcal{Z}_i - z_i)^2\Big)^{-(n+1)/2}. \tag{3.7}$$

The posterior distribution for θ can be expressed as

$$p(\theta|Y) \propto p(Y|\theta)\, p(\theta). \tag{3.8}$$

The probability of the target-rate range j is computed by integrating out the posterior uncertainty:

$$\widehat{c}_j = \int c_j\, p(c_j|Y)\, dc_j. \tag{3.9}$$

Given draws $\{\theta^{(r)}\}_{r=1}^{R}$ from the posterior [see Section 4],

$$\widehat{c}_j \approx \frac{1}{R} \sum_{r=1}^{R} c_j^{(r)}. \tag{3.10}$$

[1] $\mathsf{SM}(x|a, b, c) = a\,b\,c^{-b} x^{b-1} \left((x/c)^b + 1\right)^{-a-1}$. See the appendix for additional information.

[2] Let Δ^{J-1} denote the $(J-1)$-dimensional simplex, so that $c \in \Delta^{J-1}$ is shorthand notation for $\sum_{j=1}^{J} c_j = 1$ where $c_j \ge 0$.

[3] One may interpret λ as controling the 'hardness' of the 'soft' constraint involving the futures contract.

4. MCMC sampler

The target-range probabilities can be drawn via a Metropolis-Hastings scheme using a Dirichlet proposal:

$$q(c'|c) = \mathsf{Dirichlet}(c'|b\,c), \tag{4.1}$$

where $b > 0$ is a tuning parameter. Then

$$c^{(r+1)} = \begin{cases} c' & \mathcal{M} > w \\ c^{(r)} & \text{otherwise} \end{cases}, \tag{4.2}$$

where $w \sim \mathsf{Uniform}(0, 1)$ and

$$\mathcal{M} = \frac{p(Y|c', x, \sigma)\, p(c')}{p(Y|c^{(r)}, x, \sigma)\, p(c^{(r)})} \times \frac{q(c^{(r)}|c')}{q(c'|c^{(r)})}, \tag{4.3}$$

and where (x, σ) are the current values.

Each of the x_j can be drawn individually using a Metropolis scheme. Let x^m denote the vector with each of the components updated to its most recent value: Some of the components of x^m may have already been updated from their values in $x^{(r)}$, while other components may not have been. Let x' denote x^m with x_j^m replaced by x_j'. Let $q(x_j'|x_j) = \mathsf{N}(x_j'|x_j, s^2)$ where s is a tuning parameter. Then[4]

$$x_j^{(r+1)} = \begin{cases} x_j' & (x_{j-1}^m < x_j' < x_{j+1}^m) \text{ and } (\mathcal{M} > w) \\ x_j^m & \text{otherwise} \end{cases}, \tag{4.4}$$

where $w \sim \mathsf{Uniform}(0, 1)$ and

$$\mathcal{M} = \frac{p(Y|c, x', \sigma)\, \mathsf{N}(x_j'|\mu_j, \tau^2)}{p(Y|c, x^m, \sigma)\, \mathsf{N}(x_j^m|\mu_j, \tau^2)}. \tag{4.5}$$

When all of the components have been updated, set $x^{(r+1)} = x^m$.

The standard deviation σ can be drawn using a Metropolis scheme as well. Let $q(\sigma'|\sigma) = \mathsf{N}(\sigma'|\sigma, s^2)$, where s is a tuning parameter. Then

$$\sigma^{(r+1)} = \begin{cases} \sigma' & (\sigma' > 0) \text{ and } (\mathcal{M} > w) \\ \sigma^{(r)} & \text{otherwise} \end{cases}, \tag{4.6}$$

where

$$\mathcal{M} = \frac{p(Y|c, x, \sigma')\, p(\sigma')}{p(Y|c, x, \sigma^{(r)})\, p(\sigma^{(r)})}. \tag{4.7}$$

Model fit. Define

$$\widehat{h} := \sum_{j=1}^{J} \widehat{c}_j\, (\widehat{x}_j + \xi) \tag{4.8a}$$

$$\widehat{z}_i := \sum_{j=1}^{J} \widehat{c}_j\, B\, V(\kappa_i; \gamma_i, \widehat{x}_j + \xi, \widehat{\sigma}), \tag{4.8b}$$

[4] In (4.4) let $x_0^m = -\infty$ and $x_{J+1}^m = \infty$.

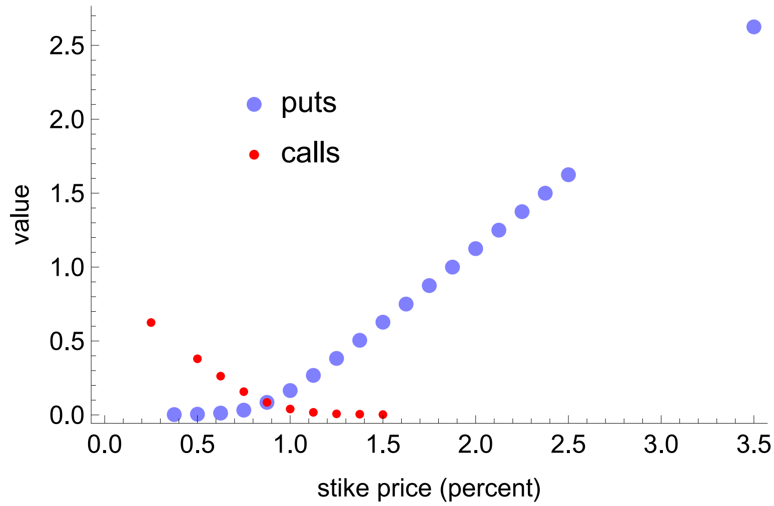

FIGURE 1. Option data as of 2016/08/03 on futures contract expiring on 2016/12/19.

where $\widehat{c}_j$ is given in (3.10) and

$$\widehat{x}_j = \frac{1}{R}\sum_{r=1}^{R} x_j^{(r)} \qquad \text{and} \qquad \widehat{\sigma} = \frac{1}{R}\sum_{r=1}^{R}\sigma^{(r)}. \tag{4.9}$$

The fit of the model can be assessed via the following (approximation to the) pricing error for the options:

$$\widehat{u} := \mathcal{H} - \widehat{h} \tag{4.10a}$$

$$\widehat{v}_i := \mathcal{Z}_i - \widehat{z}_i. \tag{4.10b}$$

5. Empirical illustration

As a first illustration, let the quote date be 2016/08/03 (`yyyy/mm/dd`). Consider the December 2016 FOMC meeting. The announcement date is 2016/12/14. The Eurodollar futures contact expires five days later on 2016/12/19.

The time to expiration is about .378 years. The discount factor is about $.999 \approx 1$. The swaps spread is about .384%. The futures rate is .875%. The option data (29 contracts[5]) are displayed in Figure 1.

Let there be $J = 8$ target-rate ranges with centers ranging from $-.125\%$ to 1.625% in increments of .25%. The values of the parameters in the prior are set as follows: $\tau = .05$, $A = .1$, and $\alpha\,\underline{c} = (1, \ldots, 1)$. The latter setting produces a flat prior for c.

See Figures 2 and 3 for plots of the posterior distribution.

[5]Explain that puts and calls are reversed from the CME because $f = 1 - P$.

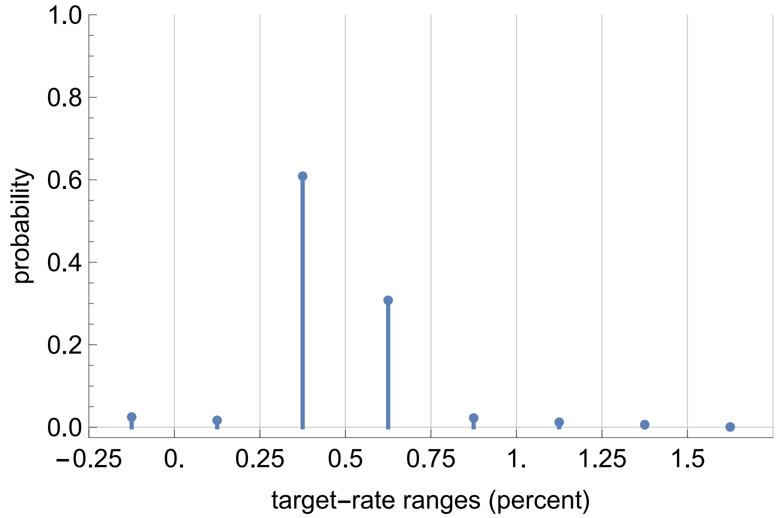

FIGURE 2. Target-rate range probabilities as of 2016/08/03 for FOMC announcement on 2016/12/14.

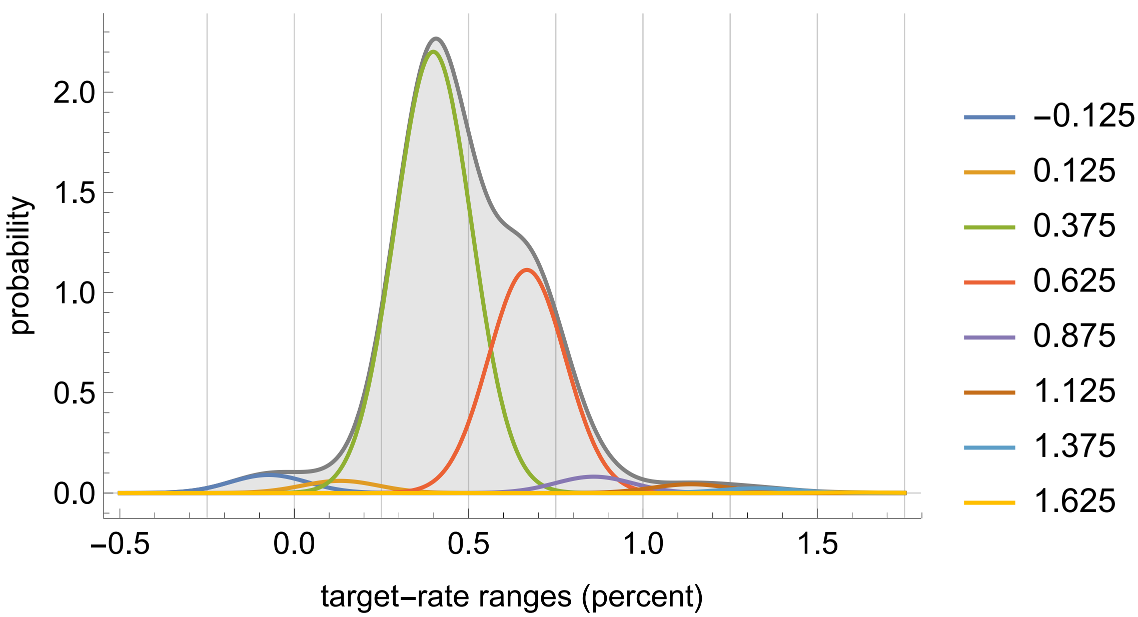

FIGURE 3. The tents associated with the probabilities shown in Figure 2. The legend indicates the centers of the included target-rate ranges.

APPENDIX A. ODDS AND ENDS

Expected option payoff. The option payoff function is given by

$$g(r;\kappa,\gamma) = \max[\kappa - r, 0] - \gamma\,(\kappa - r), \tag{A.1}$$

where κ is the strike price and $\gamma = 0$ for a put option and $\gamma = 1$ for a call option. If r is normally distributed, then the expected payoff is

$$E[g(r;\kappa,\gamma)] = \int g(r;\kappa,\gamma)\,\mathsf{N}(r|x,\sigma^2)\,dr = V(\kappa;\gamma,x,\sigma), \tag{A.2}$$

where

$$V(\kappa;\gamma,x,\sigma) := (\kappa - x)\,\Phi\!\left(\frac{\kappa - x}{\sigma}\right) + \sigma^2\,\mathsf{N}(\kappa - x|0,\sigma^2) - \gamma\,(\kappa - x), \tag{A.3}$$

and where $\Phi(\,\cdot\,)$ denotes the CDF for the standard normal distribution.

Notes:

$$V_{\kappa\kappa}(\kappa;\gamma,x,\sigma) = \mathsf{N}(\kappa|x,\sigma^2) \tag{A.4a}$$

$$\lim_{\sigma\to 0} V(\kappa;\gamma,x,\sigma) = g(x;\kappa,\gamma) \tag{A.4b}$$

$$V(\kappa;\gamma,x+\xi,\sigma) = V(\kappa-\xi;\gamma,x,\sigma). \tag{A.4c}$$

Prior for *x*. In (2.5) and (2.6), x_j is the expected future average overnight rate over the 90 days following the expiration of the futures and options contracts. On the quarterly cycle (Mar/Jun/Sep/Dec), Eurodollar futures and options contracts typically expire within a week of the day on which the FOMC announces the target-rate range for the subsequent inter-meeting period. The FOMC meets eight times per year, so the average inter-meeting period is about 45.5 days long, about half the maturity of the spot rate in question.

Let f_j denote the expected future average overnight rate on the announcement date if the announced target-rate range is j. To account for the difference in timing between the expiration of the derivatives contracts and the announcement date, let

$$x_j = f_j + \omega_j. \tag{A.5}$$

In addition, to allow for the fact that the expectation f_j involves the target-rate range to be announced at the following FOMC meeting, let

$$f_j = \mu_j + \varepsilon_j. \tag{A.6}$$

Combining (A.5) and (A.6) produces

$$x_j = \mu_j + (\varepsilon_j + \omega_j). \tag{A.7}$$

I model (A.7) as

$$p(x_j) = \mathsf{N}(x_j|\mu_j,\tau^2), \tag{A.8}$$

where τ characterizes the uncertainty regarding $\varepsilon_j + \omega_j$.

An aside. Here is some additional structure that would allow for inference about f_j if desired. Let $\varepsilon_j \sim \mathsf{N}(0,\sigma_\varepsilon^2)$ and $\omega_j \sim \mathsf{N}(0,\rho\,\sigma_\varepsilon^2)$. Then $\tau^2 = (1+\rho)\,\sigma_\varepsilon^2$ and

$$p(f_j|x_j) = \mathsf{N}(f_j|m_j,s^2), \tag{A.9}$$

where

$$m_j = x_j + \left(\frac{\rho}{1+\rho}\right)(\mu_j - x_j) \tag{A.10}$$

$$s^2 = \left(\frac{\rho}{1+\rho}\right)\sigma_\varepsilon^2. \tag{A.11}$$

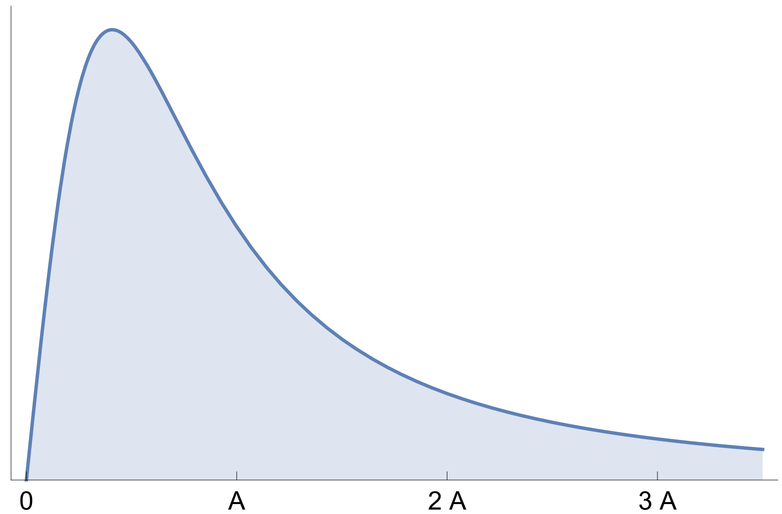

FIGURE 4. Plot of $p(\sigma) = \mathsf{SM}(\sigma|A)$.

Note that if $\rho \ll 1$, then $f_j \approx x_j$.

Singh–Maddala distribution. The distribution $\mathsf{SM}(A)$ does not have a finite mean; its median equals A. The 5th and 95th quantiles equal (approximately) $0.19\,A$ and $11.5\,A$, respectively. The mode occurs at $A/\sqrt{6} \approx 0.4\,A$. See Figure 4. An important feature of $\mathsf{SM}(A)$ is that $\mathsf{SM}(\sigma|A) \to 0$ linearly as $\sigma \to 0$.[6] In this regard, this distribution lies between distributions such as the half-Cauchy (which does not go to zero) and the inverse-gamma distribution (which goes to zero exponentially).

FEDERAL RESERVE BANK OF ATLANTA, RESEARCH DEPARTMENT, 1000 PEACHTREE STREET N.E., ATLANTA, GA 30309–4470

E-mail address: mark.fisher@atl.frb.org

URL: http://www.markfisher.net

[6] $\mathsf{SM}(\sigma|A) = (3/A^2)\,\sigma + \mathcal{O}(\sigma^3)$.